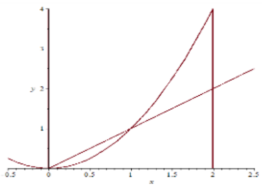
Solution to Review Questions

MAT1322D, Fall 2017

1. Find the area between the graphs of $y = x^2$ and $y = x$ in the interval [0, 2].

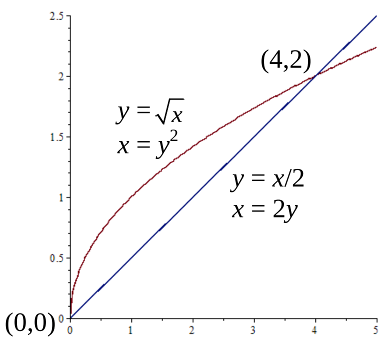
Solution. The intersection points of the graphs are (0, 0) and (1, 1). The area
$A = \int_0^1 (x - x^2)dx + \int_1^2 (x^2 - x)dx = \frac{1}{6} + \frac{5}{6} = 1.$



2. Consider the region R in x-y plane under the graph of $y = \sqrt{x}$, above the line $y = \frac{x}{2}$.

(a) If a solid has R as the base and the cross sections perpendicular to y-axis are squares, find the volume of the solid.

(b) A solid S is obtained by revolving R about the line $x = -1$. Find the volume of S.



Solution. Let $\sqrt{x} = \frac{x}{2}$. The intersections of these two curves are at (0, 0) and (4, 2).

(a) At a given value y, the side length of the cross section is $2y - y^2$. The area of the cross section is $A(y) = (2y - y^2)^2$. The volume of the solid is

$$V = \int_0^2 (2y - y^2)^2 dy = \frac{16}{15}.$$

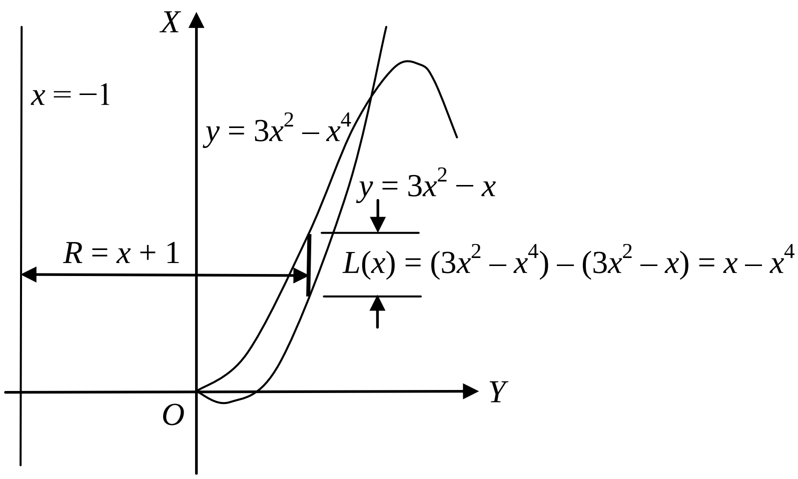
(b) Use the formula of revolving: $r_{\text{inner}} = y^2 + 1$, $r_{\text{outer}} = 2y + 1$. The volume of S is

$$V = \pi\int_0^2 \left((2y+1)^2 - (y^2+1)^2\right)dy = \pi\int_0^2 (-y^4 + 2y^2 + 4y)dy = \frac{104}{15}\pi.$$

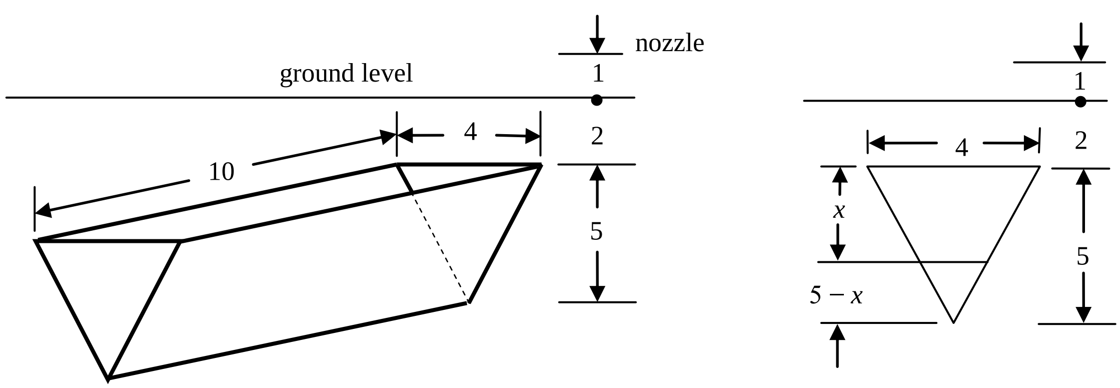
3. Let R be the region under the graph of $y = 3x^2 - x^4$ and above the graph of $y = 3x^2 - x$. Solid S is obtained by revolving region R about the axis $x = -1$. Find the volume of S.

Solution. Because the inverse function cannot be found, use the method of cylindrical shells. A vertical line segment between the graphs of $y = 3x^2 - x^4$ and the graph of $y = 3x^2 - x$ at a given value of x has length $L(x) = 3x^2 - x^4 - 3x^2 + x = x - x^4$. The distance between this line segment and the axis $x = -1$ is $R = x + 1$. The cylindrical shell obtained by revolving a vertical slice of R with width dx close to this line segment has volume $V(x) = 2\pi RL(x) = 2\pi(x + 1)(x - x^4)$. Since the intersections of these two curves are obtained by $3x^2 - x^4 = 3x^2 - x$, $x = x^4$, $x = 0$, and $x = 1$. The volume of solid S is

$$V = 2\pi\int_0^1 (x+1)(x-x^4)dx = \frac{14}{15}\pi.$$



4. A tank is of the shape of a triangular cylinder lying horizontally 2 meters under the ground as in the following figure. It is filled with oil of density ρ kg/m^3. Let g be the acceleration of gravity. Find the work, in Joules, needed to pump the oil in the tank to a nozzle 1 meter above the ground.



Solution. Look at a layer of oil x meters under the top of the tank with thickness dx. The volume of this layer is $V(x) = 10 \times \frac{4}{5}(5 - x)dx = 8(5 - x)dx$. The weight of this layer is $w(x) = \rho g dV = 8\rho g(5 - x)dx$. The work needed to pump this layer to the height of the nozzle is $dW = (x + 3)dw = 8\rho g(5 - x)(x + 3)dx$. The total work needed is

$$W = 8\rho g\int_0^5 (5-x)(x+3)dx = \frac{1400}{3}\rho g .$$

5. Find the average value of the function $y = \dfrac{x}{(x^2+1)^2}$ in the interval [1, 5].

Solution. Let $u = x^2 + 1$. Then $u' = 2x$.

The average value of y in [1, 5] is

$$\bar{y}(1,5) = \frac{1}{5-1}\int_1^5 \frac{x}{(x^2+1)^2}dx = \frac{1}{4}\int_2^{26}\frac{1}{2u^2}du = \frac{3}{52}.$$

6. Use the definition to determine whether the improper integral $\int_2^4 \frac{1}{\sqrt{x-2}}dx$ converges or diverges. If it converges, find its value.

Solution. With variable substitution $u = x - 2$, $u' = 1$,

$$\int_2^4 \frac{1}{\sqrt{x-2}}dx = \lim_{a\to 2^+}\int_a^4 \frac{1}{\sqrt{x-2}}dx = \lim_{a\to 2^+}\int_{a-2}^2 \frac{1}{\sqrt{u}}du = \lim_{a\to 2^+}\left[2\sqrt{u}\right]_{u=a-2}^2 = 2\lim_{a\to 2^+}\left(\sqrt{2}-\sqrt{a-2}\right) = 2\sqrt{2} .$$

This improper integral converges to a value $2\sqrt{2}$.

7. Use the comparison test to determine whether each of the following improper integrals is convergent or divergent:

(a) $\int_1^\infty \frac{2\sqrt{x}-1}{x+\sqrt{x}}dx$; (b) $\int_0^1 \frac{1}{2\sqrt{x}-x}dx$.

Solution. (a) When $x > 1$, $\sqrt{x} > 1$, $2\sqrt{x} - 1 = \sqrt{x}$, then $(\sqrt{x} - 1) > \sqrt{x}$. On the other hand, when $x > 1$, $\sqrt{x} < x$, then $x + \sqrt{x} < 2x$. Hence $\frac{2\sqrt{x}-1}{x+\sqrt{x}} > \frac{\sqrt{x}}{2x} = \frac{1}{2\sqrt{x}}$. Since $\int_1^\infty \frac{1}{2\sqrt{x}}dx$ diverges, $\int_1^\infty \frac{2\sqrt{x}-1}{x+\sqrt{x}}dx$ diverges.

(b) When $0 < x < 1$, $\sqrt{x} > x$, then $2\sqrt{x} - x = \sqrt{x} + (\sqrt{x} - x) > \sqrt{x}$. Hence, $\frac{1}{2\sqrt{x} - x} < \frac{1}{\sqrt{x}}$.

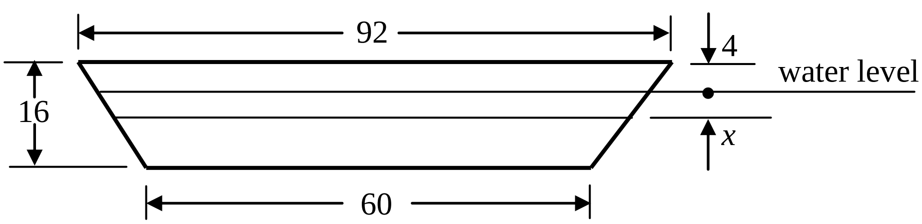
Since $\int_0^1 \frac{1}{\sqrt{x}}dx$ converges, $\int_0^1 \frac{1}{2\sqrt{x} - x}dx$ converges.

8. Find the length of the arc $y = \frac{1}{2}\ln x - \frac{1}{4}x^2$, $1 \le x \le 2$.

Solution. $y' = \frac{1}{2x} - \frac{x}{2}$. $(y')^2 = \frac{1}{4x^2} - \frac{1}{2} + \frac{x^2}{4}$. $1 + (y')^2 = \frac{1}{4x^2} + \frac{1}{2} + \frac{x^2}{4} = \left(\frac{1}{2x} + \frac{x}{2}\right)^2$.

The length of the arc is $L = \int_1^2 \sqrt{1 + (y')^2}dx = \int_1^2 \left(\frac{1}{2x} + \frac{x}{2}\right)dx = \frac{1}{2}\left[\ln x + \frac{x^2}{2}\right]_{x=1}^2 = \frac{1}{2}\left(\ln 2 + \frac{3}{2}\right)$.

9. A dam has the shape of the trapezoid shown in the following figure. The height is 16 meters and the width is 92 meters at the top, and 60 meters at the bottom. The water level is 4 meters below the top of the dam,



Let x be the depth of a horizontal stripe of the dam. Suppose ρ is the density of water, and g is the acceleration of gravity. Find the force acting on the dam, in Newtons.

Solution. The width of a stripe of the dam surface at depth x meters is $60 + 2(12 - x) = 84 - 2x$. The area of this stripe with height dx is $A(x) = (84 - 2x)dx$. The depth of this stripe is $D(x) = x$. The pressure on this stripe is $P(x) = \rho g D(x) = \rho g x$. The force acting on this stripe is $F(x) = P(x)A(x) = \rho g x(84 - 2x)\, dx$ The total force acting on the dam is

$$F = \int_0^{12} \rho g x(84 - 2x)dx \approx 4.8 \times 10^7 \text{ Newton.}$$

10. A plate takes the region $\boldsymbol{R}$ between the graph of $y = \sin x$ and the x-axis, $0 \le x \le \frac{\pi}{2}$. Assume the density is $\rho = 1$. Find the centroid of R.

Solution. The mass is $m = \int_0^{\pi/2} \sin x dx = 1$.

The moments

$M_x = \frac{1}{2}\int_0^{\pi/2} \sin^2 x dx = \frac{1}{4}\int_0^{\pi/3}(1+\cos(2x))dx = \frac{\pi}{8}$.

$M_y = \int_0^{\pi/2} x \sin x dx = 1$.

The centroid is $\left(1, \frac{\pi}{8}\right)$.

11. Use Euler's method with $h = 0.5$ to find an approximation of $y(2)$, where $y(t)$ is the solution to the initial-value problem $y' = -\frac{1}{t^3 y}$, $y(1) = 2$.

Solution. $y_{i+1} = \boldsymbol{y_i} - \frac{h}{t_i^3 y_i}$.

i	t_i	y_i
0	1	2
1	1.5	$2 - \frac{0.5}{1^3 \times 2} \approx 1.75$
2	2.0	$1.75 - \frac{0.5}{1.5^3 \times 1.75} \approx 1.67$.

12. Solve the initial-value problem $y' = y^2 \cos t$, $y(0) = 1$.

Solution. Separating variables, $\int \frac{1}{y^2} dy = \int \cos t dt$. Then $-\frac{1}{y} = \sin t + C$. By the initial condition, $C = -1$. $y = \frac{1}{1-\sin t}$.

13. The population of a certain species of fish in a lake is currently estimated 20000. Three years ago, the population was 18000. Assume the population grows according to the exponential model. What would be the population 10 years later?

Solution. The model is $P(t) = P(0)e^{kt}$. If we let the current time by $t = 0$, then $P(0) = 20000$, and $P(-3) = 18000$, and we want to find $P(10)$. Then $18000 = 20000e^{-3k}$, and $e^k = (20000 / 18000)^{1/3} = (10 / 9)^{1/3}$. $P(10) = 20000e^{10k} = 20000 \times (10 / 9)^{10/3} \approx 28415$.

14. Solve the initial-value problem $y' = 6 - y - y^2$, $y(0) = 3$.

Solution. Factorize the right-hand side and separate variables: $\int \frac{1}{(2-y)(3+y)} dy = \int dt$.

Use partial fraction:

$\dfrac{1}{(2-y)(3+y)} = \dfrac{A}{2-y} + \dfrac{B}{3+y} = \dfrac{A(3+y)+B(2-y)}{(2-y)(3+y)}$. $A(3 + y) + B(2 - y) = 1$. $A = \dfrac{1}{5}$, $B = \dfrac{1}{5}$.

Then $\displaystyle\int \frac{1}{(2-y)(3+y)}dy = \frac{1}{5}\int \left(\frac{1}{2-y} + \frac{1}{3+y}\right)dt = \frac{1}{5}\ln\left|\frac{3+y}{2-y}\right| = t + C$.

$\ln\left|\dfrac{3+y}{2-y}\right| = 5t + 5C$. $\left|\dfrac{3+y}{2-y}\right| = K_1e^{5t}$, where $K_1 = e^{5C} > 0$. Take off the absolute value sign:
$\dfrac{3+y}{2-y} = Ke^{5t}$, where $K = \pm K_1 \neq 0$.

By the initial condition, $K = -6$. Hence, $3 + y = -6e^{5t}(2 - y)$, $(1 - 6e^{5t})y = -12e^{5t} - 3$.

$y = \dfrac{12e^{5t}+3}{6e^{5t}-1}$.

15. A vat with 500 gallon of beer contains originally 4% of alcohol. Beer with 6% alcohol is pumped into this vat at a rate of 5 gallon per minute, and the mixture is pumped out at the same rate. Let $Q(t)$ be the amount of alcohol in the vat as a function of time t. Construct an initial-value problem that $Q(t)$ satisfies, and solve this equation to find $Q(t)$.

Solution. $Q(t)$ increases at a rate $r_{\text{in}} = 5 \times 0.06 = 0.3$ gallon / minute.
The concentration of alcohol in the vat at time t is $C(t) = Q(t) / 500$. $Q(t)$ decreases at a rate $r_{\text{out}} = 5 \times C(t) = 0.01Q(t)$. The net rate of change of the amount of alcohol in the vat is

$Q'(t) = 0.3 - 0.01Q(t)$.

The initial condition is $Q(0) = 500 \times 0.04 = 20$ (gallon). The equilibrium solution is $Q = \dfrac{0.3}{0.01} = 30$. The solution to this initial-value problem is not the equilibrium solution.

$\displaystyle\int \frac{1}{0.3-0.01Q}dQ = -\frac{1}{0.01}\ln|0.3-0.01Q| = \int dt = t + C$.

$\ln | 0.3 - 0.01Q | = -0.01(t + C)$. $| 0.3 - 0.01Q | = K_1e^{-0.01t}$, where $K_1 = e^{-0.01C} > 0$.

$0.3 - 0.01Q = Ke^{-0.01t}$, $K = \pm K_1 \neq 0$. With the initial condition, $K = 0.3 - 0.01 \times 20 = 0.1$.

Hence, $0.3 - 0.01Q = 0.1e^{-0.01}$. $Q = \dfrac{0.3-0.1e^{-0.01t}}{0.01} = 30 - 10e^{-0.01t}$.

16. Find the sum of the series $\sum_{n=0}^{\infty}\frac{2^{3n}-(-1)^n 5^{n+1}}{3^{2n}}$.

Solution. This series is the difference of two geometric series:

$$\sum_{n=0}^{\infty}\frac{2^{3n}-(-1)^n 5^{n+1}}{3^{2n}}=\sum_{n=0}^{\infty}\frac{2^{3n}}{3^{2n}}-\sum_{n=0}^{\infty}\frac{(-1)^n 5^{n+1}}{3^{2n}}.$$

The first series has first term $\frac{2^0}{3^0}$ = 1, and common ratio $\frac{8}{9}$. The second series has first term $\frac{5^1}{3^0}$ =5, and common ratio $-\frac{5}{9}$. The sum of the series is $S=\frac{1}{1-\frac{8}{9}}-\frac{5}{1+\frac{5}{9}}=9-\frac{45}{14}=\frac{81}{14}$.

17. Determine the convergence of the following series:

(a) $\sum_{n=2}^{\infty}\frac{1}{n(\ln n)^3}$.

Solution. Since the function $f(x)=\frac{1}{x(\ln x)^3}$ is continuous, positive, and decreasing when $x > 2$, we can use the integral test. Because the improper integral

$$\int_2^{\infty}\frac{1}{x(\ln x)^3}\,dx=\lim_{b\to\infty}\int_2^{b}\frac{1}{x(\ln x)^3}\,dx=\lim_{b\to\infty}\int_{\ln 2}^{\ln b}\frac{1}{u^3}\,du=\frac{1}{3}\lim_{b\to\infty}\left(\frac{1}{(\ln 2)^2}-\frac{1}{(\ln b)^2}\right)=\frac{1}{2(\ln 2)^2}<\infty$$

converges, this series converges.

(b) $\sum_{n=0}^{\infty}\frac{2n-1}{n\sqrt{n}+1}$.

Solution. This is a positive series when $n \geq 1$. (The first term is negative, which does not affect the convergence of the series.) Use limit comparison test.

Let $a_n=\frac{2n-1}{n\sqrt{n}+1}$, and let $b_n=\frac{1}{\sqrt{n}}$.

$\lim_{n\to\infty}\frac{a_n}{b_n}=\lim_{n\to\infty}\frac{(2n-1)\sqrt{n}}{n\sqrt{n}+1}=\lim_{n\to\infty}\frac{2n\sqrt{n}-\sqrt{n}}{n\sqrt{n}+1}=\lim_{n\to\infty}\frac{2-1/n}{1+1/n^{3/2}}=2$. Since $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ diverges. This series diverges.

(c) $\displaystyle\sum_{n=0}^{\infty}\frac{n+\sin^2 n}{2n^2\sqrt{n}-1}$.

Solution. Use comparison test. Since, when $n > 1$, $n + \sin^2 n < 2n$, and $2n^2\sqrt{n} - 1 = n^2\sqrt{n} + (n^2\sqrt{n} - 1) > n^2\sqrt{n}$, $\dfrac{n+\sin^2 n}{2n^2\sqrt{n}-1} < \dfrac{2n}{n^2\sqrt{n}} = \dfrac{2}{n^{3/2}}$. Since $\displaystyle\sum_{n=1}^{\infty}\frac{2}{n^{3/2}} = 2\sum_{n=1}^{\infty}\frac{1}{n^{3/2}}$ converges, this series converges.

(d) $\displaystyle\sum_{n=2}^{\infty}(-1)^n\frac{n+\sin n}{3n+1}$.

Solution. Since $\displaystyle\lim_{n\to\infty}\frac{n+\sin n}{3n+1} = \lim_{n\to\infty}\frac{n}{3n+1} + \lim_{n\to\infty}\frac{\sin n}{3n+1} = \frac{1}{3} + 0 = \frac{1}{3} \neq 0$, this series is divergent.

(e) $\displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{\sqrt{n+1}}{3n+1}$.

Solution. Since $f(n) = \dfrac{\sqrt{n+1}}{3n+1}$ is decreasing and approaches zero when n approaches infinity, by the alternating series test, this series is convergent.

(f) $\displaystyle\sum_{n=1}^{\infty}\left(\frac{2^n+1}{2^{n+1}+1}\right)^n$.

Solution. Use the root test. Let $a_n = \left(\dfrac{2^n+1}{2^{n+1}+1}\right)^n$. Then $\sqrt[n]{a_n} = \dfrac{2^n+1}{2^{n+1}+1}$.

$\displaystyle\lim_{n\to\infty}\frac{2^n+1}{2^{n+1}+1} = \lim_{n\to\infty}\frac{1+1/2^n}{2+1/2^n} = \frac{1}{2}$. This series is convergent.

18. $S_{10} = \displaystyle\sum_{n=2}^{10}\frac{1}{n(\ln n)^3} \approx 1.9755$. Find an upper bound and a lower bound of the sum of the series $S = \displaystyle\sum_{n=2}^{\infty}\frac{1}{n(\ln n)^3}$.

Solution. Let $u = \ln x$. $\displaystyle\int \frac{1}{x(\ln x)^3}\,dx = -\frac{1}{2(\ln x)^2} + C$. Hence,

$\displaystyle\int_{10}^{\infty}\frac{1}{x(\ln x)^3}\,dx = -\lim_{b\to\infty}\left[\frac{1}{2(\ln x)^2}\right]_{x=10}^{b} = \frac{1}{2(\ln 10)^2} \approx 0.0943$, and

$$\int_{11}^{\infty} \frac{1}{x(\ln x)^3}\,dx = -\lim_{b\to\infty}\left[\frac{1}{2(\ln x)^2}\right]_{x=11}^{b} = \frac{1}{2(\ln 11)^2} \approx 0.0870.$$

Hence, 1.9755 + 0.0870 = 2.0625 < S < 1.9755 + 0.0943 = 2.0698.

19. The partial sum $S_{10} = \sum_{n=0}^{10}(-1)^n \frac{\sqrt{n+1}}{3n+1} \approx 0.8375$. Find an upper bound and a lower bound of the sum of the series $S = \sum_{n=0}^{\infty}(-1)^n \frac{\sqrt{n+1}}{3n+1}$.

Solution. Since $a_{11} = (-1)^{11}\frac{\sqrt{11+1}}{3\times 11+1} \approx -0.1019 < 0$, S_{10} is an overestimate, and

$S_{10} + a_{11} = 0.7356 < S < 0.8475$.

20. Consider power series $\sum_{n=1}^{\infty}\frac{(x+3)^n}{2^{2n}\sqrt{n}}$. For which value(s) of x, is this series absolutely convergent? For which value(s) of x, is this series conditionally convergent? For which value(s) of x, is this series divergent?

Find the radius of convergence and the interval of convergence of this series.

Solution. The center of the series is $x = -3$. Use the ratio test.

$$\lim_{n\to 0}\left|\frac{(x+3)^{n+1}}{2^{2n+2}\sqrt{n+1}}\frac{2^{2n}\sqrt{n}}{(x+3)^n}\right| = \frac{1}{4}\lim_{n\to 0}\left|\sqrt{\frac{n}{n+1}}(x+3)\right| = \frac{1}{4}|x+3|.$$

When $|x+3| < 4$, $-7 < x < 1$, this series is absolutely convergent.

When $|x+3| > 4$, $x < -7$ or $x > 1$, this series is divergent.

When $x = -7$, this series becomes $\sum_{n=1}^{\infty}\frac{(-7+3)^n}{2^{2n}\sqrt{n}} = \sum_{n=1}^{\infty}\frac{(-4)^n}{2^{2n}\sqrt{n}} = \sum_{n=1}^{\infty}(-1)^n\frac{1}{\sqrt{n}}$. By alternating series test, it is convergent. Since $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ is divergent, this series is conditionally convergent at $x = -7$.

When $x = 1$, this series becomes $\sum_{n=1}^{\infty}\frac{(1+3)^n}{2^{2n}\sqrt{n}} = \sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$. This is a p-series with $p = 1$. It is divergent.

The radius of convergence of this series is 4, and the interval of convergence of this series is

$[-7, 1)$.

21. Consider the function $y = f(x)$, defined by $f(x) = \int_0^x \frac{1}{1+2t^4}\,dt$.

(a) Find the Maclaurin series of this function.

(b) Find $\frac{d^9}{dx^9} f(0)$ and $\frac{d^{10}}{dx^{10}} f(0)$.

Solution. (a) $\frac{1}{1-r} = 1 + r + r^2 + r^3 + \ldots$. Let $r = -2t^4$. $\frac{1}{1+2t^4} = 1 - 2t^4 + 2^2t^8 - 2^3t^{12} + \ldots$.

$$\int_0^x \frac{1}{1+2t^4}\,dt = \int_0^x \left(1 - 2t^4 + 2^2t^8 - 2^3t^{12} + \ldots\right)dx = x - \frac{2x^5}{5} + \frac{4x^9}{9} - \frac{8x^{13}}{13} + \ldots \ .$$

(b) $\frac{d^9}{dx^9} f(0) = \frac{4}{9}9! = 161280$. $\frac{d^{10}}{dx^{10}} f(0) = 0$.

22. (a) Find the first four non-zero terms of the Maclaurin series of the function $y = \frac{1}{\sqrt{1+x^2}}$.

(b) Recall that $\frac{d}{dx}\ln(x + \sqrt{1+x^2}) = \frac{1}{\sqrt{1+x^2}}$. Find the first four non-zero terms of the Maclaurin series of the function $y = \ln(x + \sqrt{1+x^2})$.

Solution. (a) By the binomial series,

$$\frac{1}{\sqrt{1+t}} = (1+t)^{-1/2} = 1 - \frac{1}{2}t + \frac{(-1/2)(-3/2)}{2!}t^2 + \frac{(-1/2)(-3/2)(-5/2)}{3!}t^3 + \ldots$$
$$= 1 - \frac{1}{2}t + \frac{3}{8}t^2 - \frac{5}{16}t^3 + \ldots .$$

Let $t = x^2$. We have $\frac{1}{\sqrt{1+x^2}} = 1 - \frac{1}{2}x^2 + \frac{3}{8}x^4 - \frac{5}{16}x^6 + \ldots$.

(b) $\ln(x + \sqrt{1+x^2}) = \int_0^x \frac{1}{\sqrt{1+t^2}}\,dt = x - \frac{1}{6}x^3 + \frac{3}{40}x^5 - \frac{5}{112}x^7 + \ldots$.

23. Find the second partial derivatives z_x, z_y, z_{xx}, z_{yy}, and z_{xy}, of the function $z = x\ln(2y - x)$.

Solution. $z_x = \ln(2y-x) - \dfrac{x}{2y-x}$. $z_{xx} = -\dfrac{1}{2y-x} - \dfrac{(2y-x)+x}{(2y-x)^2} = \dfrac{-2y+x-2y}{(2y-x)^2} = \dfrac{x-4y}{(2y-x)^2}$.

$z_y = \dfrac{2x}{2y-x}$. $z_{yy} = -\dfrac{4x}{(2y-x)^2}$. $z_{xy} = z_{yx} = \dfrac{4y}{(2y-x)^2}$.

24. Consider function $z = x^2y - 3x^2 + y^2$.

(a) Find the partial derivatives z_x, z_y, z_{xx}, z_{yy}, and z_{xy}.

(b) Find the gradient vector of z at $x = -1$, and $y = 2$.

(c) Find the directional derivative of z at the point $(-1, 2)$ in the direction of $\mathbf{v} = (4, -3)$.

(d) Find the equation of the tangent plane of the graph of this function at the point where $x = 1$, and $y = 2$.

Solution. (a) $z_x = 2xy - 6x$, $z_y = x^2 + 2y$, $z_{xx} = 2y - 6$, $z_{yy} = 2$, $z_{xy} = 2x$.

(b) The gradient vector of z is grad $z(x, y) = (2xy - 6x, x^2 + 2y)$. At point $(-1, 2)$, grad $z(-1, 2) = (2, 5)$.

(c) The unit vector in the direction of $\mathbf{v}$ is $\mathbf{u} = \left(\dfrac{4}{5}, -\dfrac{3}{5}\right)$. The directional derivative is

$$D_{\mathbf{v}}(z) = \left(\frac{4}{5}, -\frac{3}{5}\right)\cdot(2,5) = \frac{8}{5} - 3 = -\frac{7}{5}.$$

(d) $z(-1, 2) = 3$. The equation of the tangent plane of the graph of this function at $x = -1$ and $y = 2$ is

$z = 2(x + 1) + 5(y - 2) + 3$, or $2x + 5y - z = 5$.

25. Let $z = f(x, y)$, where $x = g(u, v)$ and $y = h(u, v)$. Then $z = f(g(u, v), h(u, v)) = F(u, v)$ is a function of u and v. Suppose the following values are known:

$f(1, 2) = 3$, $f_x(1, 2) = 5$, $f_y(1, 2) = -2$;
$g(3, 5) = 1$, $g_u(3, 5) = 2$, $g_v(3, 5) = -1$;
$h(3, 5) = 2$, $h_u(3, 5) = 4$, $h_v(3, 5) = -3$.

Find F_u and F_v at $u = 3$, $v = 5$.

Solution. $F_u(3, 5) = f_x(3, 5)\, g_u(1, 2) + f_v(3, 5) h_u(1, 2) = 5 \times 2 + (-2) \times 4 = 2$, and

$F_v(3, 5) = f_x(3, 5)\, g_v(1, 2) + f_y(3, 5) h_v(1, 2) = 5 \times (-1) + (-2) \times (-3) = 1$.

26. Consider function $z = f(x, y)$ defined implicitly by the equation $x^2z + xy - yz^3 = -1$.

(a) Find the gradient vector of the function $z = f(x, y)$ at the point (2, 1, −1).

(b) Find the equation of the tangent plane of the graph of the equation at the point (2, 1, −1).

(c) Find the directional derivative of this function in the direction of the vector $\mathbf{u} = (2, -3)$.

(d) Find the maximum value of the directional derivative at (2, 1, −1) among all possible directions.

Solution. Let $F(x, y, z) = x^2z + xy - yz^3 + 1$. $F_x = 2xz + y$, $F_y = x - z^3$, $F_z = x^2 - 3yz^2$.
$F_x(2, 1, -1) = -3$, $F_y(2, 1, -1) = 3$, $F_z(2, -1, 3) = 1$. Then $z_x = 3$ and $z_y = 3$. The gradient vector is $\nabla f(2, 1) = (3, -3)$.

(b) The equation of the tangent plane of the graph of this equation at the point (2, 1, −1) is $-3(x - 2) + 3(y - 1) + (z + 1) = 0$, or $-3x + 3y + z + 4 = 0$.

(c) The unit vector in the direction of $\mathbf{u}$ is $\mathbf{v} = \dfrac{\mathbf{u}}{|\mathbf{u}|} = \left(\dfrac{2}{\sqrt{13}}, -\dfrac{3}{\sqrt{13}}\right)$. The directional derivative is $D_{\mathbf{u}}(f) = \nabla f \cdot \mathbf{v} = 3\times\dfrac{2}{\sqrt{13}} - 3\times\dfrac{3}{\sqrt{13}} = -\dfrac{3}{\sqrt{13}}$.

(d) The maximum derivative is the length of the gradient vector $\|(3, -3)\| = 3\sqrt{2}$.